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On the Fundamental Gaps of Some Saturated Numerical Semigroups with Multiplicity 4

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Abstract

In this study, we calculate the number of fundamental gaps of the some numerical semigroups S which are $S = <4, k, k+2, k+3>$ for $k \equiv 3 \pmod 4$ and $k \geq 7$, and $S = <4, k, k+t, k+t+2>$ for $k \equiv 2 \pmod 4$ and $k \geq 6$, and $t = 1$ or $t = 3$. Also, we give the type sequence of these numerical semigroups.

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1. Introduction

Throughout this paper, we assume that $\mathbb{N}$ and $\mathbb{Z}$ be the sets of nonnegative integers and integers, respectively. The subset S of $\mathbb{N}$ is a numerical semigroup if $0 \in S$, $x+y \in S, for\ all\ x, y \in S$, and $Card(\mathbb{N} \backslash S) < \infty$ (this condition is equivalent to $\gcd(S)=1$, $\gcd(S)=$ greatest common divisor the element of S) (See [5]).

A numerical semigroup S can examine that

$$S=<x_1, x_2, ..., x_v>=\left\{\sum_{i=1}^{v} n_i a_i \ : n_i \in \mathbb{N}\right\}$$

where $x_1, x_2, ..., x_v \in S$ and $v \geq 1$. If there isn't any subset A of $\{x_1, x_2, ..., x_v\}$ such that $<A>=S$ then the set $\{x_1, x_2, ..., x_v\}$ is minimal system of generators of S. So, we called that v is embedding dimension of S, and we denote by $e(S)=v$ .Also, $m(S)=\min\{s \in S : s>0\}$ is called multiplicity of S. In general, the inequally $e(S) \leq m(S)$ holds. We called that S has maximal embedding dimension if $e(S)=m(S)$.

Let S be a numerical semigroup, then $F(S)=\max(\mathbb{Z} \backslash S)$ is called Frobenius number of S. Also, $n(S)=Card(\{0,1,2,...,F(S)\} \cap S)$ is called the number determine of S.

If S is a numerical semigroup such that $S=<x_1, x_2, ..., x_v>$, then we observe that

$$S=<x_1, x_2, ..., x_v>=\{s_0=0, s_, , s_2, ..., s_{n-1}, s_n=F(S)+1, \rightarrow ...\} ,$$

where $s_i < s_{i+1}$, $n=n(S)$, and the arrow means that every integer greater than $F(S)+1$ belongs to S, for $i=1,2,...,n=n(S)$. (See [6]).

If $a \in \mathbb{N}$ and $a \notin S$, then a is called gap of S. We denote the set of gaps of S, by $H(S)$, i.e, $H(S)=\mathbb{N} \backslash S$ .The $g(S)=Card(H(S))$ is called the genus of S. We called that y is a fundamental gap of S, if $y \in H(S)$ and $\{2y, 3y\} \subset S$. It is denoted by $FH(S)$ the set of all the fundamental gaps of S, i.e.

$FH(S)=\{y \in H(S) : \{2y, 3y\} \subset S\}$. (See [2]).

Let $S=\{s_0=0,s_1,s_2,...,s_{n-1},s_n=F(S)+1,\rightarrow...\}$ be a numerical semigroup, where $n=n(S)$ and $s_i<s_{i+1}$. Then we define sets S_i and $S(i)$ as follows (for $i=0,1,2,...,n(S)$):

$$S_i=\{s\in S: s\geq s_i\} \text{ and } S(i)=S-S_i=\{\, a\in\mathbb{N}: a+S_i\subseteq S \,\}.$$

So, each $S(i)$ is itself a numerical semigroup and we write that the following inclusion chain:

$$S_n\subset S_{n-1}\subset...\subset S_1\subset S_0=S=S(0)\subset S(1)\subset...\subset S(n-1)\subset S(n)=\mathbb{N}.$$

The number $t(S)=Card(S(1)\backslash S)$ is called the type of S and $t_i(S)=Card(S(i)\setminus S(i-1))$, for $i\geq 1$. İt is clear that $t_1(S)=t(S)$, but, in the general case, $t_i(S)\neq t(S(i))$. The set $\{t_1,t_2,...,t_{n(S)}\}$ is called the type sequence of S.(See [3]).

A numerical semigroup S is Arf if $a+b-c\in S$, for all $a,b,c\in S$ such that $a\geq b\geq c$. It known that an Arf numerical semigroup has maximal embedding dimension. On the other hand, Arf numerical semigroup S satisfy property $t_i=s_i-s_{i-1}-1$, for $1\leq i\leq n=n(S)$.

A numerical semigroup S is Saturated if $s+d_1s_1+d_2s_2+...+d_ms_m\in S$, where $s,s_i\in S$ and $d_i\in\mathbb{Z}$ such that $d_1s_1+d_2s_2+...+d_ms_m\geq 0$ and $s_i\leq s$ for $i=1,2,...,m$. A Saturated numerical is Arf, but an Arf numerical semigroup needn't be Saturated. (See [1]).

In this work, we calculate the number of fundamental gaps of the some numerical semigroup S which are $S=<4,k,k+2,k+3>$, for $k\equiv 3(\text{mod}\,4)$ and $k\geq 7$, and $S=<4,k,k+t,k+t+2>$, for $k\equiv 2(\text{mod}\,4)$ and $k\geq 6$, and $t=1$ or $t=3$. Also, we give the type sequence these numerical semigroups.

2. Main Results

Lemma 2.1 Let S be a numerical semigroup which is one of each $S=<4,k,k+2,k+3>$ (for $k\equiv 3(\text{mod}\,4)$ and $k\geq 7$) and $S=<4,k,k+t,k+t+2>$ (for $k\equiv 2(\text{mod}\,4)$ and $k\geq 10$, and t an odd integer). Then, the following are true:

(a) If $x \in H(S)$ then $\frac{x}{2} \notin FH(S)$.

(b) If $x \in H(S)$ then $\frac{x}{3} \notin FH(S)$.

(c) If $x = 1,2,3$ then $x \notin FH(S)$.

(d) If $x = 2F(S)$ then $x \in S$.

Proof. (a) Let $x \in H(S)$. Then,

(i) If x is odd integer then $\frac{x}{2} \notin FH(S)$ because of $\frac{x}{2} \notin \mathbb{N}$.

(ii) If x is even integer then $\frac{x}{2} \notin FH(S)$ because of $2(\frac{x}{2}) = x \in S$.

(b) Conversely, we assume that $\frac{x}{3} \in FH(S)$ when $x \in H(S)$. Then, we have $3(\frac{x}{3}) = x \in S$. But, this is a contradiction. In this case, we write that $\frac{x}{3} \notin FH(S)$ if $x \in H(S)$.

(c) If the numerical semigroup $S = < 4, k, k+2, k+3 >$ (for $k \equiv 3 \pmod 4$ and $k \geq 7$) then, it is obvious $x \notin FH(S)$ for $x = 1,2,3$. If the numerical semigroup $S = < 4, k, k+t, k+t+2 >$ (for $k \equiv 2 \pmod 4$ and $k \geq 6$, and t an odd integer), then $x \in H(S)$, for $x = 1,2,3$. then $x \notin FH(S)$.

(d) If $x = 2F(S)$ then $x \in S$ because of $x = 2F(S) > F(S)+1 \in S$.

Theorem 2.2 [4] Let $S = < 4, k, k+2, k+3 >$ be numerical semigroup, where $k \in \mathbb{Z}$, $k \equiv 3 \pmod 4$ and $k \geq 7$. Then, we have following equalities:

(a) $F(S) = k-1$

(b) $g(S) = \frac{3k-1}{4}$

(c) $PF(S) = \{ k-4, k-2, k-1 \}$

(d) $n(S) = \frac{k+1}{4}$

(e) $H(S) = \{ 1,2,3,5,6,7,...,k-6, k-5, k-4, k-2, k-1 \}$.

Proposition 2.3 Let $S=<4,k,k+2,k+3>$ be a numerical semigroup, where for $k\equiv 3(\mathrm{mod}\,4)$ and $k\geq 7$. Then, we have

(a) $FH(S)=\{k-1,k-2,k-4,k-5,k-6,...,6,5\}$.

(b) The sequence type of S is $\{t_1=3,\ t_2=3,....,t_{n-1}=3,\ t_n=2\}$.

Proof. Let $S=<4,k,k+2,k+3>$ be a numerical semigroup, where for $k\equiv 3(\mathrm{mod}\,4)$ and $k\geq 7$.

(a) Then, the set of gaps of S is

$$H(S)=\{1,2,3,5,6,7,...,k-6,k-5,k-4,k-2,k-1\}$$

from Theorem 2.2. Thus we find that the set of fundamental gaps of S is $FH(S)=\{k-1,k-2,k-4,k-5,k-6,...,6,5\}$ by Lemma 2.1.

(b) Then, we write that

$$S=<4,k,k+2,k+3>=\{s_0=0,s_1=4,s_2=8,...,s_{n-2}=k-7,s_{n-1}=k-3,\ s_n=k\}$$

and S is saturated numerical semigroup. So, we have that $t_i=s_i-s_{i-1}-1$ because S is Arf. In this case, we find that

$t_1=s_1-s_0-1=3,\ t_2=s_2-s_1-1=3,...,t_{n-1}=s_{n-1}-s_{n-2}-1=(k-3)-(k-7)-1=3$ and $t_n=s_n-s_{n-1}-1=k-(k-3)-1=2$.

Corollary 2.4 Let $S=<4,k,k+2,k+3>$ be a numerical semigroup, where for $k\equiv 3(\mathrm{mod}\,4)$ and $k\geq 7$. If $12p+7\leq k\leq 12p+15$, for $p=0,1,2,....$ then $Card(FH(S))=\dfrac{k-3}{2}-p$.

Example 2.5 (a) We consider $S=<4,7,9,10>=\{0,4,7,\rightarrow...\}$ numerical semigroup. Then, we have $F(S)=6$, $n(S)=2, H(S)=\{1,2,3,5,6\}$ and $FH(S)=\{5,6\}$. So, we find that $Card\,FH(S)=\dfrac{7-3}{2}-0=2$ (for $k=7$ and $p=0$). Also, we write that the sequence type of S is $\{t_1=3,\ t_2=2\}$

(b) we consider the numerical semigroup $S=<4,19,21,22>=\{0,4,8,12,16,19\rightarrow...\}$. Then, we have

$$F(S)=18,\ n(S)=5, H(S)=\{1,2,3,5,6,7,9,10,11,13,14,15,17,18\}$$

and $FH(S)=\{18,17,15,14,13,11,10\}$. So, we find that $Card\,FH(S)=\dfrac{19-3}{2}-1=7$ (for $k=19$ and $p=1$). Also, we have that the sequence type of S is $\{t_1=t_2=...=t_6=3,t_7=2\}$.

Proposition 2.6 Let $S=<4,k,k+t,k+t+2>$ be numerical semigroup, where $k\equiv 2(\mathrm{mod}\,4)$ and $k\geq 6$, and t is an odd integer. Then, it satisfies the following conditions:

(a) If $t=1$ then

(i) $FH(S)=\{\,k-1,k-3,k-4,k-5,...,6,5\,\}$.
(ii) The sequence type of S is $\{\,t_1=3,\ t_2=3,....,t_{n-1}=3,\ t_n=1\,\}$.

(b) If $t=3$ then

(i) $FH(S)=\{\,k+1,k-1,k-3,k-4,k-5,...,6,5\,\}$.
(ii) The sequence type of S is $\{\,t_1=3,\ t_2=3,...t_{n-2}=3,\ t_{n-1}=1,\ t_n=1\,\}$.

Proof. Let $S=<4,k,k+t,k+t+2>$ be numerical semigroup, where $k\equiv 2\,(\text{mod}\,4)$ and $k\geq 6$, and t is an odd integer. Then, it satisfies the following conditions:

(a) If $t=1$ then we observe that

$$S=<4,k,k+1,k+3>=\{0,4,8,...,k-6,k-2,k,\rightarrow ...\}$$

and

$$H(S)=\{1,2,3,5,6,7,...,k-5,k-4,k-3,k-1\,\}.$$

So, we have that
(i) $FH(S)=\{\,k-1,k-3,k-4,k-5,...,6,5\,\}$ from Lemma 2.1.
(ii) We write $t_i=s_i-s_{i-1}-1$ since S is Arf. In this case, we find that
$t_1=s_1-s_0-1=3,\ t_2=s_2-s_1-1=3,...,t_{n-2}=3,$ and
$t_{n-1}=s_{n-1}-s_{n-2}-1=(k-2)-(k-6)-1=3.,\ t_n=s_n-s_{n-1}-1=k-(k-2)-1=1.$

(b) If $t=3$ then we observe that

$$S=<4,k,k+3,k+5>=\{0,4,8,...,k-2,k,k+2,k+4,\rightarrow ...\}$$

and

$$H(S)=\{1,2,3,5,6,7,...,k-5,k-4,k-3,k-1,k+1,k+3\,\}.$$

So, we have that
(i) $FH(S)=\{\,k+1,k-1,k-3,k-4,k-5,...,6,5\,\}$ from Lemma 2.1.
(ii) We write $t_i=s_i-s_{i-1}-1$ because S is Arf. In this case, we find that
$t_1=s_1-s_0-1=3,\ t_2=s_2-s_1-1=3,...,t_{n-2}=3,$ and
$t_{n-1}=s_{n-1}-s_{n-2}-1=(k+2)-(k)-1=1,\ t_n=s_n-s_{n-1}-1=(k+4)-(k+2)-1=1.$

Corollary 2.7 Let $S=<4,k,k+t,k+t+2>$ be numerical semigroup, where $k\equiv 2\,(\text{mod}\,4)$ and $k\geq 6$, and t is an odd integer. Then, it satisfies the following conditions:

(a) If $t=1$ and $12p+10\leq k\leq 12p+18$, for $p=0,1,2,...$ then $Card(FH(S))=\dfrac{k-2}{2}-p$.

(b) If $t=3$ and $12p+10 \le k \le 12p+18$, for $p=0,1,2,...$ then $Card(FH(S)) = \frac{k-2}{2} - (p-1)$.

Example 2.8

(a) We consider the numerical semigroup

$$S = < 4,14,15,17 > = \{0,4,8,12,14,\rightarrow ...\}.$$

Then, we have
$F(S)=13$, $n(S)=4, H(S)=\{1,2,3,5,6,7,9,10,11,13\}$ and $FH(S)=\{6,7,9,10,11,13\}$.
So, we find that $Card\,FH(S) = \frac{14-2}{2} - 0 = 6$ (for $k=14, t=1$ and $p=0$). Also, we write that the sequence type of S is $\{\, t_1 = t_2 = t_3 = 3,\ t_4 = 1 \,\}$.

(b) We consider the numerical semigroup

$$S = < 4,34,35,37 > = \{0,4,8,12,16,20,24,28,32,34,\rightarrow ...\}.$$

Then, we have

$F(S)=33$, $n(S)=9$,
$H(S)=\{1,2,3,5,6,7,9,10,11,13,14,15,17,18,19,21,22,23,25,26,27,29,30,31,33\}$
and

$$FH(S)=\{14,17,18,19,21,22,23,25,26,27,29,30,31,33\}.$$

So, we find that $Card\,FH(S) = \frac{34-2}{2} - 2 = 14$ (for $k=34$, $t=1$ and $p=2$). Also, we have that the sequence type of S is $\{\, t_1 = t_2 = ... = t_8 = 3,\ t_9 = 1 \,\}$.

Example 2.19 (a) We consider the numerical semigroup

$$S = < 4,10,13,15 > = \{0,4,8,10,12,\rightarrow ...\}.$$

Then, we have $F(S)=11$, $n(S)=4, H(S)=\{1,2,3,5,6,7,9,11\}$ and $FH(S)=\{5,6,7,9,11\}$. So, we find that $Card\,FH(S) = \frac{10-2}{2} - (0-1) = 5$ (for $k=10$, $t=3$ and $p=0$). Also, we write that the sequence type of S is $\{\, t_1 = t_2 = 3,\ t_3 = t_4 = 1 \,\}$.

(b) We consider the numerical semigroup

$S = < 4,34,37,39 > = \{0,4,8,12,16,20,24,28,32,34,36,\rightarrow ...\}$.
Then, we have

$F(S)=35$, $n(S)=10$,
$H(S)=\{1,2,3,5,6,7,9,10,11,13,14,15,17,18,19,21,22,23,25,26,27,29,30,31,33,35\}$
and

$$FH(S)=\{14,17,18,19,21,22,23,25,26,27,29,30,31,33,35\}.$$

So, we find that $Card\,FH(S)=\frac{34-2}{2}-(2-1)=15$ (for $k=34$, $t=3$ and $p=2$). Also, we write that the sequence type of S is $\{t_1=t_2=...=t_8=3,\ t_9=t_{10}=1\}$.

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