Selçuk J. Appl. Math.
Vol. 13. No. 1. pp. 69-74, 2012

Selçuk Journal of Applied Mathematics

Some Extensions of a Class of Pseudo Symmetric Numerical Semigroups

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Received Date: September 24, 2011
Accepted Date: May 11, 2012

Abstract. In this paper, we will give some results about some extensions of a pseudo symmetric numerical semigroup in the form of $S =< 3,\ 3+s,\ 3+2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

Key words: Numerical semigroup; Pseudo symmetric; Extension.
2000 Mathematics Subject Classification: 20M14.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers respectively. A numerical semigroup is a subset S of $\mathbb{N}$ that is closed under addition where $0 \in S$ and $\mathbb{N}\backslash S$ is finite. It is well known that every numerical semigroup is finitely generated [1], that is to say, there exist $s_1, s_2, ..., s_p \in \mathbb{N}$ such that $s_1 < s_2 < ... < s_p$ and

$$S =< s_1, s_2, ..., s_p >= \{s_1k_1 + s_2k_2 + ... + s_pk_p : k_i \in \mathbb{N}, 1 \le i \le p\}.$$

Moreover, every numerical semigroup has a unique minimal system of generators. In this case, we say that $\mu(S) = \min\{s \in S\ :\ s > 0\}$ is the multiplicity of S, and $e(S) = \sharp(\{s_1, s_2, ..., s_p\})$ is the embedding dimension of S. S has maximal embedding dimension if $\mu(S) = e(S)$.
Following the notation used in [2,3], if S is a numerical semigroup then the greatest integer in $\mathbb{Z}\backslash S$ is the *Frobenius number* of S , denoted by $g(S)$. The elements of $\mathbb{N}\backslash S$, denoted by $H(S)$ are called *gaps* of S.

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S is *symmetric* if for every $x \in \mathbb{Z} \backslash S$, the integer $g(S) - x \in S$. Similarly, S is *pseudo symmetric* if $g(S)$ is even and there exists an integer $x \in \mathbb{Z} \backslash S$ such that $x = \frac{g(S)}{2}$ and $g(S) - x \notin S$. For more background on symmetric and pseudo symmetric numerical semigroups, the reader is encouraged to see [2,3,4,7,9].
Let S be a numerical semigroup and $m \in S \backslash \{0\}$. The Apery set of S with respect to m is defined by $Ap(S,\ m) = \{s \in S :\ s - m \notin S\}$. Furthermore, it is known that S has maximal embedding dimension if and only if $Ap(S,\ m) = \{0, s_2, s_3, ...s_p\}$ by [9]. Hence, $Ap(S,\ m) = \{w(0) = 0, w(1), w(2), ..., w(m-1)\}$ and $g(S) = \max(Ap(S, m)) - m$, where $w(i)$ is the least element in S that is congruent with i modulo m. For instance see [6] and [10].
The following can be found in [7]: Let S be a numerical semigroup. We say that $x \in \mathbb{Z} \backslash S$ is a *pseudo Frobenius number* of S if $x + s \in S$ for all $s \in S \backslash \{0\}$. We denote by $Pg(S)$ the set of pseudo Frobenius numbers of S. The cardinal of $Pg(S)$ is called the type of S and denoted by *type*(S). Notice that $g(S)$ is always an element of $Pg(S)$. In [11], it is proved that a numerical semigroup is symmetric if and only if $Pg(S) = \{g(S)\}$ i.e. *type*$(S) = 1$. Furthermore, we define in S the following partial order:

$$a \leq_S b \text{ if } b - a \in S.$$

For $m \in S \backslash \{0\}$, it is proved that $Pg(S) = \{w(i) - m :\ w(i)$ *maximals* $\leq_S Ap(S, m)\}$ in [7].
An element $x \in Pg(S)$ is a *special gap* of S if $2x \in S$. We denote by $SH(S)$ the set of special gaps of S. That is, $SH(S) = \{x \in Pg(S) :\ 2x \in S\}$.
The main goal of this paper, is to give some extensions of a pseudo symmetric numerical semigroup in the form of $S =< 3,\ 3 + s,\ 3 + 2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$. In this extension, any numerical semigroup is determined by Corollary 2.4. Some of these are symmetric, some are pseudo symmetric and some are neither symmetric nor pseudo symmetric numerical semigroups. Furthermore, the extensions of S is characterized by Theorem 2.1 which are pseudo symmetric numerical semigroups.
In this paper, S is defined as $S =< 3,\ 3 + s,\ 3 + 2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

2. Results

In this section, we will give some results about some extensions of a pseudo symmetric numerical semigroup in the form $S =< 3,\ 3 + s,\ 3 + 2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

Firstly, we shall give the following result which is given by Lemma 8 and Lemma 9 in [4]:

Corollary 2.1. *If* $S =< 3,\ 3 + s,\ 3 + 2s >$ *then* $\sharp(H(S)) = s + 1$, *where* $s \in \mathbb{Z}^+$ *and* $3 \nmid s$.

Now we shall give Lemma 2.1 and Proposition 2.1 in [8]:

Lemma 2.1. *For a numerical S, we can recurrently define S_n as follows:*

(1) $S_0 = S$

(2) $S_{n+1} = \begin{cases} S_n \cup \{g(S_n)\} & ;\text{ if } S_n \neq \mathbb{N} \\ \mathbb{N} & ;\text{ in other cases} \end{cases}$

If $k = \sharp(H(S))$, then it is clear that

$$S = S_0 \subset S_1 \subset S_2 \subset ... \subset S_{s-1} =< 3,4,5 >\subset S_s =< 2,3 >\subset S_k = \mathbb{N}.$$

Proposition 2.1. *Let S be a numerical semigroup and $x \in H(S)$. The following properties are equivalent:*

(1) $x \in SH(S)$

(2) $S \cup \{x\}$ is a numerical semigroup.

Corollary 2.2. $S \cup \{2s\}$ *is a numerical semigroup.*

Proposition 2.2. *If* $S_1 = S \cup \{2s\}$ *then* $S_1 =< 3, 3+s, 2s >$.

Proof. It is easy to show that $< 3, 3+s, 2s >=< 3, 3+s, 2s, 2s+3 >\supseteq< 3, 3+s, 2s+3 > \cup\{2s\}$. Conversely, let $x \in S_1 = S \cup \{2s\}$. In this case, either $x \in< 3, 3+s, 2s+3 >$ or $x \in \{2s\}$. If $x \in< 3, 3+s, 2s+3 >$ then we have $x = 3k_1 + (3+s)k_2 + (2s+3)k_3,\ k_1, k_2, k_3 \in \mathbb{N}$. Here, we find that $x = 3(k_1 + k_2) + (3+s)k_2 + 2sk_3 \in< 3, 3+s, 2s >$. If $x \in \{2s\}$ then it is clear that $x \in< 3, 3+s, 2s >$.

Lemma 2.2. *Let* $S_1 =< 3, 3+s, 2s >$. *Then* $S_1 \cup \{s\}$ *is a numerical semigroup.*

Proof. S_1 has maximal embedding dimension, since $e(S) = \mu(S) = 3$. Thus, we have $Ap(S_1, 3) = \{0, 3+s, 2s\}$ by [9].In this case, we write $maximals_{\leq_S}$ $(Ap(S_1, 3)) = \{3+s, 2s\}$. Therefore, we find that $Pg(S_1) = \{w - 3 : w \in \{3+s, 2s\}\} = \{s, 2s-3\}$ and $SH(S_1) = \{x \in Pg(S_1) : 2x \in S_1\} = \{s, 2s-3\}$. Hence, we obtain that $S_1 \cup \{s\}$ is a numerical semigroup, by Proposition 2.1.

Proposition 2.3. *If* $S_2 =< 3, 3+s, 2s > \cup \{s\}$ *then* $S_2 =< 3, s >$.

Proof. It is clear that $< 3, s >=< 3, 3+s, 2s >\subseteq< 3, 3+s, 2s > \cup \{s\}$. On the other hand, let us $x \in< 3, 3+s, 2s > \cup \{s\}$. In this case, either $x \in< 3, 3+s, 2s >$ or $x \in \{s\}$. If $x \in< 3, 3+s, 2s >$ then we have $x = 3k_1 + (3+s)k_2 +$

$(2s)k_3,\ k_1,k_2,k_3 \in \mathbb{N}$. Here, we find that $x = 3(k_1+k_2)+s(k_2+2k_3) \in< 3, s >$. If $x \in \{2s\}$ then it is trivial that $x \in< 3, s >$.

Lemma 2.3. Let $S_2 =< 3, s >$, where $s \geq 4$, $s \in \mathbb{Z}^+$ and $3 \nmid s$. Then $S_2 \cup \{2s-3\}$ is a numerical semigroup.

Proof. The proof is similar to the one in Lemma 2.2.

Proposition 2.4. *If* $S_3 =< 3, s > \cup \{2s-3\}$ *, where* $s \geq 4$*,* $s \in \mathbb{Z}^+$ *and* $3 \nmid s$*, then* $S_3 =< 3, s, 2s-3 >$.

Proof. It is clear that $< 3, s, 2s-3 >=< 3, s, 2s-3, 2s, s+3 >=< 3, s >\subset< 3, s > \cup \{s\}$. On the other hand, let $x \in< 3, s > \cup \{2s-3\}$. In this case, either $x \in< 3, s >$ or $x \in \{2s-3\}$. If $x \in< 3, s >$ then we have $x = 3k_1 + sk_2 + (2s-3).0,\ k_1, k_2 \in \mathbb{N}$. Here, we find that $x \in< 3, s, 2s-3 >$. If $x \in \{2s-3\}$ then it is trivial that $x \in< 3, s, 2s-3 >$.

The following corollary is a result of Corollary 2.1:

Corollary 2.3. *Let* S *be a pseudo symmetric numerical semigroup in the form* $S =< 3,\ 3+s,\ 3+2s >$ *for* $s \in \mathbb{Z}^+$ *and* $3 \nmid s$*. Then we have the following increasing chain of extension of* S *, by Lemma 2.1*:

$$S = S_0 \subset S_1 \subset S_2 \subset ... \subset S_s =< 2, 3 >\subset S_{s+1} = \mathbb{N}.$$

Corollary 2.4. *For* $0 \leq k \leq s+1$*, any numerical semigroup* S_k *in the above chain is given as follows:*

$$S_k = \begin{cases} < 3, 3+(s-k), 3+2(s-k) > & \text{if } k \equiv 0(\text{ modulo } 3) \\ < 3, 3+(s-k+1), 2(s-k+1) > & \text{if } k \equiv 1(\text{ modulo } 3) \\ < 3, s-k+2 > & \text{if } k \equiv 2(\text{ modulo } 3) \end{cases}$$

Theorem 2.1. *Let* S_k *be any numerical semigroup in the extension chain of numerical semigroups of* S*.for* $0 \leq k \leq s-1$ *. If* $k \equiv 0($ *modulo 3*$)$ *then* S_k *is pseudo symmetric.*

Proof. (a) If $k = 0$ then we have $S_0 =< 3, 3+s, 2s+3 >= S$ by Corollary 2.4.

(b) Let $k = 3r$ where $r > 0$ and $r \in \mathbb{Z}$. We shall prove the theorem by induction on r:

The Apery set of $S_3 =< 3, s, 2s-3 >$ is given by $Ap(S_3, 3) = \{0, s, 2s-3\}$. In fact,

$w(0) = 0$, and $w(1) = s$ since $3 \nmid s$, $w(2) = 2s-3$, since $2s-3 \equiv 2$ (*modulo 3*). Therefore, $Ap(S_3, 3) = \{w(0) = 0, w(1) = s, w(2) = 2s-3\}$. On the other

hand,
(i) If $r=1$ then $S_k=S_3=<3,s,2s-3>$ is pseudo symmetric :

$$g(S_3)=\max(Ap(S_3,3))-3=2s-6.$$

Thus, we have $Ap(S_3,3)=\{w(0)=0,w(1),w(2)\}=\{s\}\cup\{2s-3\}=\left\{\frac{2(s-3)}{2}+3\right\}\cup$ $\{0,2s-3\}$. Hence, we find that S_3 is pseudo symmetric by [3].

(ii) We assume that $S_k=S_{3n}=<3,3+(s-3n),3+2(s-3n)>$ is pseudo symmetric for $r=n$.

(iii) Finally, we show that $S_k=S_{3(n+1)}$ is pseudo symmetric:

$$\begin{aligned} S_k &= S_{3(n+1)}=<3,3+(s-3n-3),3+2(s-3n-3)> \\ &= <3,s-3n,2(s-3n)-3>. \end{aligned}$$

In this case, we put $t=s-3n$. Then, $3\nmid t$ since $3\nmid s$.
Thus, we find that $S_k=S_{3(n+1)}=<3,t,2t-3>$ is pseudo symmetric by (i).

Example 2.1. *Let* $S=<3,8,13>=\{0,3,6,8,9,11,\rightarrow...\}$ *be a pseudo symmetric numerical semigroup for* $s=5$. *Then,* $g(S)=10$, $Ap(S,3)=\{0,8,13\}$, *and* $H(S)=\{1,2,4,5,7,10\}$ *and* $SH(S)=\{10\}$. *Thus,* $S_1=S\cup\{10\}=\{0,3,6,8,9,10,11,\rightarrow...\}=<3,8,10>$, $g(S_1)=7$ *and* $SH(S_1)=\{5,7\}=Pg(S_1)$.

In this case, we have extensions of S as follows:

$S_2=S_1\cup\{5\}=<3,5>$, $S_3=S_2\cup\{7\}=<3,5,7>$,
$S_4=<3,5,4>=<3,4,5>$, $S_5=<2,3>=\mathbb{N}\backslash\{1\}$ and $S_6=\mathbb{N}$.

Thus, we obtaine the following chain:

$S=S_0=<3,8,13>\subset S_1=<3,8,10>\subset S_2=<3,5>\subset S_3=<3,5,7>\subset S_4=<3,4,5>\subset S_5=<2,3>=\mathbb{N}\backslash\{1\}\subset S_6=\mathbb{N}$.

In this extension chain, S_0,S_3 and S_4 are pseudo symmetric by Corollary 2.4. and Theorem 2.1. However, S_2,S_5 and S_6 are symmetric numerical semigroups by Corollary 2.4. But, S_1 is neither a symmetric nor a pseudo numerical semigroup.

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